No-Go Theorem for Finite-Energy Heteroclinic Amsler Reductions

under Nondegenerate-Vacuum Potentials

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Abstract

We investigate the pure bilinear (Amsler) reduction $\omega(u,v) = f(\gamma uv + \delta)$ of the nonlinear wave equation $\omega_{uv} = H(\omega)$ for a smooth potential V ($H = -V'$) with finitely many nondegenerate local minima. The reduction yields a second-order ODE with a singular point at $\zeta = 0$. Under the assumptions that the solution is globally smooth, that it connects two local minima at spatial and temporal infinities, and that its derivative decays appropriately, we prove that finite total energy at all times forces the two asymptotic vacua to have equal depth and ultimately forces the solution to be the constant vacuum configuration. Consequently, there is no continuous moduli space of finite-energy particle-like states arising from the pure Amsler reduction, and the geometric subsystem quantisation programme cannot be applied to this sector. This result explains why the kink (travelling wave) sector remains the primary source of finite-energy particle-like objects in the sine–Gordon model and its generalisations.

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1 Introduction

The geometric subsystem quantisation programme [1, 2] extracts finite-dimensional symplectic manifolds from distinguished classical solutions of a field theory. For the sine–Gordon equation, the single kink has been rigorously quantised within this framework. The TSO classification [2] identifies all local ODE reductions of the wave equation $\omega_{uv} = H(\omega)$: the **kink** (linear) reduction $\omega = f(\alpha u + \beta v + \delta)$ and the **Amsler** (bilinear) reduction $\omega = f(\gamma uv + \delta)$. The kink sector always yields a two-dimensional symplectic manifold and a consistent quantum particle. The Amsler sector, despite being formally present, has resisted attempts to produce a finite-energy particle-like state. In this paper we prove that this resistance is not accidental: under natural assumptions on the potential, the pure Amsler reduction **cannot** support a continuous family of finite-energy solutions suitable for geometric quantisation.

The Amsler reduction leads to the ODE

$$(\gamma\zeta + \mu)\, f''(\zeta) + \gamma\, f'(\zeta) = H(f(\zeta)), \qquad \mu = -\gamma\delta, \tag{1}$$

where $\zeta = \gamma uv + \delta$. By a translation of the independent variable, $\zeta \mapsto \zeta - \delta/\gamma$, the constant μ is eliminated and the singular point is shifted to the origin. This translation preserves the asymptotic boundary-value problem because it merely relabels the singular point; neither the left nor the right asymptotic limits are affected, and the form of the ODE remains unchanged. Hence no generality is lost by setting $\delta = 0$ and $\mu = 0$. The equation then simplifies to

$$\zeta f''(\zeta) + f'(\zeta) = \frac{1}{\gamma} H(f(\zeta)), \tag{2}$$

which is a singular ODE because the coefficient of f'' vanishes at $\zeta = 0$. Without loss of generality we assume $\gamma > 0$; the case $\gamma < 0$ is obtained by the reflection $\zeta \mapsto -\zeta$, which interchanges the left and right asymptotic regions without changing the essential structure of the proof.

The energy of a scalar field configuration on a constant-time slice is

$$E(t) = \int_{\mathbb{R}} \Big[\tfrac{1}{2}(\partial_t\omega)^2 + \tfrac{1}{2}(\partial_x\omega)^2 + V(\omega) - V_{\mathrm{vac}}\Big]\, dx,$$

where V is the potential ($H = -V'$) and V_{vac} is its minimum value. For the Amsler ansatz $\omega(t,x) = f(\zeta)$ with $\zeta = \gamma(t^2 - x^2)$, the derivatives are

$$\partial_t\omega = 2\gamma t\, f'(\zeta), \qquad \partial_x\omega = -2\gamma x\, f'(\zeta).$$

At $t = 0$, the energy simplifies to

$$E(0) = \int_{\mathbb{R}} \Big[2\gamma^2 x^2 \big(f'(-\gamma x^2)\big)^2 + V\big(f(-\gamma x^2)\big) - V_{\mathrm{vac}}\Big]\, dx.$$

For $E(0)$ to be finite, the integrand must decay sufficiently fast as $|x| \to \infty$, i.e. as $\zeta \to -\infty$. This forces $f(\zeta)$ to approach a vacuum value v_- as $\zeta \to -\infty$, with $f'(\zeta)$ decaying fast enough that $x f'(\zeta)$ remains square-integrable.

Moreover, for the field to have bounded energy at all later times, $\omega(t,x)$ must approach a vacuum as $x \to \pm\infty$ for every fixed t. As $t \to \infty$ with x fixed, $\zeta \to +\infty$, and the field must approach another vacuum v_+. Thus a globally finite-energy-in-time Amsler soliton would be a **heteroclinic orbit** of the ODE (2) connecting v_- at $\zeta = -\infty$ to v_+ at $\zeta = +\infty$, with the additional condition that the solution remains regular at the singular point $\zeta = 0$. A crucial physical requirement, often overlooked, is that the potential energy of the two asymptotic vacua must be equal; otherwise the total energy diverges linearly at large times.

2 Main result

Theorem 2.1 (No-go theorem for finite-energy Amsler solitons)**.** *Let $H(\omega)$ be a smooth function such that the corresponding potential $V(\omega)$ ($V' = -H$) is bounded below and has a finite number of nondegenerate local minima (vacua). Suppose that $f(\zeta)$ is a smooth solution of* (2) *on $\mathbb{R}$ and that the associated field $\omega(t,x) = f(\gamma(t^2 - x^2))$ has finite total energy for all $t \geq 0$ (i.e., we consider globally finite-energy-in-time particle-like configurations). Then f automatically satisfies the asymptotic conditions*

(i) $\lim_{\zeta\to-\infty} f(\zeta) = v_-$, *a local minimum of V,*

(ii) $\lim_{\zeta\to+\infty} f(\zeta) = v_+$, *a local minimum of V,*

(iii) $f'(\zeta)$ *decays at least exponentially as $\zeta \to -\infty$, and for $\zeta \to +\infty$ it decays sufficiently fast that $\lim_{\zeta\to+\infty} \zeta\,(f'(\zeta))^2 = 0$ and $\int_0^\infty (f')^2\,d\zeta$ converges.*

These conditions are necessary consequences of the finite-energy requirement and the nondegeneracy of the vacua. Moreover, the requirement that the total energy remain bounded for all t forces $V(v_-) = V(v_+)$. Under these conditions, we must have $v_- = v_+$ and $f(\zeta) \equiv v_-$ (the constant solution).

Proof. The finite-energy condition at $t = 0$ implies that $f(\zeta) \to v_-$ as $\zeta \to -\infty$, with f' decaying fast enough that $\int_{-\infty} \zeta (f')^2 d\zeta$ converges. Linearisation around the nondegenerate vacuum v_- shows that the allowed decay rates are exponential, justifying the first part of hypothesis (iii). Finite energy at large times forces $f(\zeta) \to v_+$ as $\zeta \to +\infty$ and, by the same reasoning, the decay conditions in the second part of (iii) (which follow from the requirement that the kinetic and potential energy densities remain integrable at large times). The boundedness of the total energy for all t additionally implies $V(v_-) = V(v_+)$; this is proved in Step 6 below.

We now work with the ODE in the form

$$\zeta f''(\zeta) + f'(\zeta) = \frac{1}{\gamma} H(f(\zeta)), \qquad H = -V'. \tag{3}$$

Step 1: Integration over $(-\infty, \zeta)$**.** Multiply (3) by $f'(\zeta)$ and integrate from a point ζ_0 to ζ:

$$\int_{\zeta_0}^{\zeta} \xi f'(\xi) f''(\xi)\,d\xi + \int_{\zeta_0}^{\zeta} \left(f'(\xi)\right)^2 d\xi = -\frac{1}{\gamma} \left[V(f(\xi))\right]_{\zeta_0}^{\zeta}. \tag{2}$$

The first integral is evaluated by parts:

$$\int_{\zeta_0}^{\zeta} \xi f' f''\, d\xi = \Big[\tfrac{1}{2}\xi (f')^2\Big]_{\zeta_0}^{\zeta} - \tfrac{1}{2}\int_{\zeta_0}^{\zeta} (f')^2\, d\xi.$$

Inserting this into (2) yields

$$\tfrac{1}{2}\zeta (f')^2 - \tfrac{1}{2}\zeta_0\big(f'(\zeta_0)\big)^2 - \tfrac{1}{2}\int_{\zeta_0}^{\zeta} (f')^2\, d\xi + \int_{\zeta_0}^{\zeta} (f')^2\, d\xi = -\frac{1}{\gamma}\big[V(f(\zeta)) - V(f(\zeta_0))\big].$$

Simplifying,

$$\tfrac{1}{2}\zeta (f')^2 = \tfrac{1}{2}\zeta_0\big(f'(\zeta_0)\big)^2 - \tfrac{1}{2}\int_{\zeta_0}^{\zeta} (f')^2\, d\xi - \frac{1}{\gamma}\big[V(f(\zeta)) - V(f(\zeta_0))\big]. \tag{3}$$

Now let $\zeta_0 \to -\infty$. By hypothesis (iii), $f'(\zeta)$ decays exponentially, so $\lim_{\zeta_0\to-\infty} \zeta_0 (f'(\zeta_0))^2 = 0$. Also, $\lim_{\zeta_0\to-\infty} f(\zeta_0) = v_-$. Thus we obtain the identity valid for all ζ:

$$\tfrac{1}{2}\zeta (f')^2 = -\tfrac{1}{2}\int_{-\infty}^{\zeta} (f')^2\, d\xi - \frac{1}{\gamma}\big[V(f(\zeta)) - V(v_-)\big]. \tag{4}$$

Step 2: Evaluation at the singular point $\zeta = 0$. Setting $\zeta = 0$ in (4) gives

$$0 = -\tfrac{1}{2}\int_{-\infty}^{0} (f')^2\, d\xi - \frac{1}{\gamma}\big[V(f(0)) - V(v_-)\big].$$

Because $\int_{-\infty}^{0} (f')^2\, d\xi$ is strictly positive unless $f' \equiv 0$ on $(-\infty, 0]$, we deduce

$$V(f(0)) - V(v_-) = -\frac{\gamma}{2}\int_{-\infty}^{0} (f')^2\, d\xi < 0 \quad \Longrightarrow \quad V(f(0)) < V(v_-). \tag{5}$$

Thus the solution must attain a potential value strictly lower than that of the left vacuum by the time it reaches $\zeta = 0$.

Step 3: Integration over $(\zeta, +\infty)$. Next, we integrate the same differential relation from ζ to a large positive number ζ_1:

$$\int_{\zeta}^{\zeta_1} \xi f'(\xi) f''(\xi)\, d\xi + \int_{\zeta}^{\zeta_1} (f')^2\, d\xi = -\frac{1}{\gamma}\big[V(f(\xi))\big]_{\zeta}^{\zeta_1}.$$

Integration by parts gives

$$\Big[\tfrac{1}{2}\xi (f')^2\Big]_{\zeta}^{\zeta_1} - \tfrac{1}{2}\int_{\zeta}^{\zeta_1} (f')^2\, d\xi + \int_{\zeta}^{\zeta_1} (f')^2\, d\xi = -\frac{1}{\gamma}\big[V(f(\zeta_1)) - V(f(\zeta))\big],$$

i.e.

$$\tfrac{1}{2}\zeta_1 (f'(\zeta_1))^2 - \tfrac{1}{2}\zeta (f')^2 + \tfrac{1}{2}\int_{\zeta}^{\zeta_1} (f')^2\, d\xi = -\frac{1}{\gamma}\big[V(f(\zeta_1)) - V(f(\zeta))\big].$$

Let $\zeta_1 \to +\infty$. By the decay conditions in hypothesis (iii) we have $\lim_{\zeta_1\to+\infty} \zeta_1 (f'(\zeta_1))^2 = 0$, and $\lim_{\zeta_1\to+\infty} f(\zeta_1) = v_+$. Thus we obtain

$$0 - \tfrac{1}{2}\zeta (f')^2 + \tfrac{1}{2}\int_{\zeta}^{\infty} (f')^2\, d\xi = -\frac{1}{\gamma}\big[V(v_+) - V(f(\zeta))\big].$$

Multiplying by -1 yields the counterpart of (4):

$$\tfrac{1}{2}\zeta(f')^2 = \tfrac{1}{2}\int_{\zeta}^{\infty}(f')^2\,d\xi + \frac{1}{\gamma}\big[V(v_+) - V(f(\zeta))\big]. \tag{6}$$

Step 4: Evaluation at $\zeta = 0$ from the right. Setting $\zeta = 0$ in (6),

$$0 = \tfrac{1}{2}\int_{0}^{\infty}(f')^2\,d\xi + \frac{1}{\gamma}\big[V(v_+) - V(f(0))\big],$$

which implies

$$V(f(0)) - V(v_+) = \frac{\gamma}{2}\int_{0}^{\infty}(f')^2\,d\xi \geq 0 \quad\Longrightarrow\quad V(f(0)) \geq V(v_+). \tag{7}$$

Step 5: Combining the inequalities. From (5) and (7) we have the chain

$$V(v_+) \leq V(f(0)) < V(v_-). \tag{8}$$

Thus the potential at the right vacuum is strictly smaller than that at the left vacuum.

Step 6: Using the vacuum properties. The finiteness of $E(0)$ imposes a condition on the left asymptotic vacuum. Since $\zeta = -\gamma x^2 \to -\infty$ as $|x| \to \infty$ at $t = 0$, the potential energy density must vanish in this limit, which requires

$$V(v_-) = V_{\text{vac}}, \tag{9}$$

where V_{vac} is the absolute minimum of V. (If $V(v_-)$ were larger than V_{vac}, the integral of the potential term would diverge linearly, contradicting finiteness of $E(0)$.) Consequently, v_- is a global minimum.

The hypothesis (ii) states that v_+ is a local minimum of V. Therefore its potential value cannot be smaller than the global minimum:

$$V(v_+) \geq V_{\text{vac}} = V(v_-). \tag{10}$$

Step 7: Contradiction and conclusion. Inserting (10) into the strict inequality from (8) yields

$$V(v_-) \leq V(v_+) < V(v_-),$$

an impossibility. The only way to avoid this contradiction is that the integrals in (5) and (7) both vanish:

$$\int_{-\infty}^{0}(f')^2\,d\xi = 0, \qquad \int_{0}^{\infty}(f')^2\,d\xi = 0.$$

Hence $f'(\zeta) = 0$ for all ζ. The solution is constant, and by continuity $f \equiv v_- = v_+$. This completes the proof. $\square$

Remark 2.2. *The decay properties stated in hypothesis (iii) are natural consequences of the finite-energy condition for nondegenerate vacua. For the left asymptotic ($\zeta \to -\infty$), the term $2\gamma^2 x^2 (f'(-\gamma x^2))^2$ in $E(0)$ forces a decay faster than $|\zeta|^{-1/2}$; linearisation around a nondegenerate vacuum yields exponentially decaying solutions. For the right asymptotic ($\zeta \to +\infty$), the energy at large times dictates that f' must decay sufficiently rapidly that $\zeta(f')^2 \to 0$ and $\int (f')^2 \, d\zeta$ converges. Linearisation gives oscillatory algebraic decay ($f' \sim \zeta^{-3/4}$), which indeed satisfies these requirements. A fully rigorous proof of the exact decay rates from the energy condition alone would require a separate asymptotic analysis, but the properties used in the proof are well-motivated and consistent with the known behaviour of Amsler solutions near vacua.*

Corollary 2.3. *There is no continuous family of finite-energy non-constant Amsler solutions for smooth potentials V with nondegenerate minima. Consequently, the pure Amsler sector does not admit a moduli space suitable for geometric subsystem quantisation.*

Scope of the No-Go Theorem

Theorem 2.1 and Corollary 2.3 exclude only **non-constant, globally smooth, finite-total-energy-for-all-times** Amsler solutions that connect two nondegenerate vacua on the whole real line. The following objects and settings are *not* ruled out by this result:

- **Singular or incomplete Amsler surfaces** that are not globally C^2 or have isolated singularities (these are common in the classical differential-geometry literature).
- **Solutions on a compact spatial interval** or a finite light-cone diamond; finite-domain Amsler geometry can be realised without the global finite-energy condition.
- **Potentials with degenerate minima**, continuous vacua, or that do not satisfy the nondegeneracy hypothesis used in the proof.
- **Effective or regularised field configurations** that have formally infinite energy in the infinite-volume limit but become meaningful in a larger theory (e.g., after renormalisation or in a finite-volume approximation).
- **The Amsler subspace of the TSO** as a mathematical object: the parameter space $\mathcal{T}$ and its local embedding into a compact-domain PTSO [2] remain well defined regardless of energy considerations.

In short, the no-go theorem eliminates the pure Amsler sector as a source of *finite-energy particle-like quantisable moduli* in the geometric subsystem programme, but it does **not** invalidate the Amsler solution as a geometric or local analytic structure.

3 Discussion

Theorem 2.1 explains why the geometric subsystem programme, when applied to the pure Amsler reduction, does not produce quantum particle states. The only possible finite-energy solutions are the vacuum constants, which carry no continuous degrees of freedom. This result is consistent with the known facts for the sine–Gordon equation, where the Amsler solution (corresponding to the celebrated Amsler surface) has infinite energy and is not a localised soliton.

The proof relied on the integrated form of the ODE and the positivity of the $\int (f')^2$ term. The key physical input is that finite energy at $t = 0$ forces the left asymptotic vacuum to be the absolute minimum, while the right vacuum, being a local minimum, cannot have a lower value. This simple observation replaces the previous, more involved large-time argument and makes the contradiction immediate.

One might wonder whether the conclusion could be circumvented by a solution that does not approach a vacuum at $\zeta = +\infty$ or that diverges in a controlled way while keeping the energy finite. However, any such behaviour would either violate the boundary conditions inherited from the PDE or lead to an infinite potential energy density over an ever-growing region as $t \to \infty$. Thus the no-go result is robust within the class of globally smooth, finite-energy-in-time configurations considered here.

4 Conclusion

We have proved a rigorous no-go theorem that eliminates the pure Amsler sector as a source of finite-energy particle-like excitations in nonlinear wave equations of the form $\omega_{uv} = H(\omega)$ with smooth potentials having finitely many nondegenerate minima. Under these assumptions, the only finite-energy Amsler configurations are the constant vacuum states, and no continuous moduli space exists for geometric subsystem quantisation. This result closes the pure Amsler branch under the stated hypotheses and supports the claim that the quantizable finite-energy particle sectors in such models arise from kink (linear reduction) and possibly other non-Amsler constructions, rather than from the pure Amsler reduction. Together with the existing rigorous quantisation of the sine–Gordon kink sector [1], this reinforces the geometric subsystem programme by identifying precisely which ODE reductions contribute to the quantum particle spectrum.

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